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FINITE TYPE INVARIANTS OF STRING LINKS AND THE HOMFLYPT POLYNOMIAL OF KNOTS

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ABSTRACT. A family of finite type invariants of string link is given by the HOMFLYPT polynomial of knots using various closure operations on (cabled) string links. In this note we will show the following:
(1) These invariants, together with Milnor invariants of length ≤ 5, give classifications of n-string links up to C_k-equivalence for $k \leq 5$, and give a complete set of finite type invariants of degree ≤ 4.
(2) Any Milnor invariant of length $n+1(>2)$ of a C_n-trivial string link is expressed as a linear combination of such invariants.

1. STRING LINKS AND C_k-MOVES

The notion of string link was introduced by Le Dimet [3] and Habegger-Lin [5]. A string link is a kind of tangle without closed components in the cylinder, which generalizes pure braids.

Let D be the unit disk in the plane. Choose n points $p_1, ..., p_n$ in the interior of D so that $p_1, ..., p_n$ lie in order on the x-axis, see Figure 2.1. An *n-string link* $L = K_1 \cup \cdots \cup K_n$ in $D \times [0,1]$ is a disjoint union of oriented arcs $K_1, ..., K_n$ such that each K_i runs from $(p_i, 0)$ to $(p_i, 1)$ $(i = 1, ..., n)$. The string link $K_1 \cup \cdots \cup K_n$ with $K_i = \{p_i\} \times [0,1]$ $(i = 1, ..., n)$ is called the *trivial n-string link* and denoted by $\mathbf{1}_n$.

The set $\mathcal{SL}(n)$ of isotopy classes of n-string links fixing the endpoints has a monoidal structure, with composition given by the *stacking product* and with the trivial n-string link $\mathbf{1}_n$ as unit element.

Habiro [6] and Goussarov [4] introduced independently the notion of *C_k-move* as follows. (This notion can also be defined by using the theory of claspers, see Subsection 5.1.) A C_k-move is a local move on (string) links as illustrated in Figure 1.1, which can be regarded as a kind of 'higher order crossing change' (in particular, a C_1-move is a crossing change). The C_k-move generates an equivalence relation on (string) links, called *C_k-equivalence*, which becomes finer as k increases. Thus we have a descending filtration

$$\mathcal{SL}(n) = \mathcal{SL}_1(n) \supset \mathcal{SL}_2(n) \supset \mathcal{SL}_3(n) \supset ...$$

where $\mathcal{SL}_k(n)$ denotes the set of *C_k-trivial* n-string links, i.e., string links which are C_k-equivalent to $\mathbf{1}_n$. For $1 \leq k \leq l$, let $\mathcal{SL}_k(n)/C_l$ denote the set of C_l-equivalence

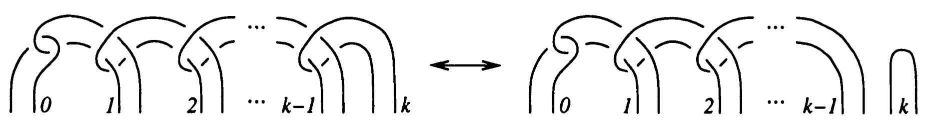

FIGURE 1.1. A C_k-move involves $k+1$ strands of a link, labelled here by integers between 0 and k.

classes of C_k-trivial n-string links. This is known to be a finitely generated nilpotent group. Furthermore, if $l \leq 2k$, this group is abelian [6, Thm. 5.4].

2. Finite type invariants of string links

A *singular n-string links* is a proper immersion $\sqcup_{i=1}^{n}[0,1]_i \to D^2 \times [0,1]$ of the disjoint union $\sqcup_{i=1}^{n}[0,1]_i$ of n copies of $[0,1]$ in $D^2 \times [0,1]$ such that the image of $[0,1]_i$ runs from $(p_i, 0)$ to $(p_i, 1)$ $(1 \leq i \leq n)$, and whose singularities are transverse double points (in finite number).

Denote by $\mathbf{Z}\mathcal{SL}(n)$ the free abelian group generated by $\mathcal{SL}(n)$. A singular n-string link σ with k double points can be expressed as an element of $\mathbf{Z}\mathcal{SL}(n)$ using the following skein formula.

(2.1) 

Let A be an abelian group. An n-string link invariant $f : \mathcal{SL}(n) \to A$ is a *finite type invariant of degree* $\leq k$ if its linear extension to $\mathbf{Z}\mathcal{SL}(n)$ vanishes on every n-string-link with (at least) $k+1$ double points. If f is of degree $\leq k$ but not of degree $k-1$, then f is called a finite type invariant of degree k.

We recall a few classical examples of such invariants in the next two subsections.

2.1. **Finite type knot invariants.** Recall that the *Conway polynomial* of a knot K has the form

$$\nabla_K(z) = 1 + \sum_{k \geq 1} a_{2k}(K) z^{2k}.$$

It is not hard to show that the z^{2k}-coefficient a_{2k} in the Conway polynomial is a finite type invariant of degree $2k$ [1].

Recall also that the *HOMFLYPT polynomial* of a knot K is of the form

$$P(K; t, z) = \sum_{k=0}^{N} P_{2k}(K; t) z^{2k},$$

where $P_{2k}(K; t) \in \mathbb{Z}[t^{\pm 1}]$ is called the $2k$th coefficient polynomial of K. Denote by $P_{2k}^{(l)}(K)$ the lth derivative of $P_{2k}(K; t)$ evaluated at $t = 1$. It was proved by Kanenobu and Miyazawa that $P_{2k}^{(l)}$ is a finite type invariant of degree $2k + l$ [9].

Note that both the Conway and HOMFLYPT polynomials of knots are invariant under orientation reversal, and that both are multiplicative under the connected sum [12].

2.2. **Milnor invariants of string links.** Given an n-component oriented, ordered link L in S^3, Milnor invariants $\overline{\mu}_L(I)$ of L are defined for each multi-index $I = i_1 i_2 ... i_m$ (i.e., any sequence of possibly repeating indices) among $\{1, ..., n\}$ [18, 19]. The number m is called the *length* of Milnor invariant $\overline{\mu}(I)$, and is denoted by $|I|$. Unfortunately, the definition of these $\overline{\mu}(I)$ contains a rather intricate self-recurrent indeterminacy.

Habegger and Lin showed that Milnor invariants are actually well defined integer-valued invariants of *string* links [5], and that the indeterminacy in Milnor invariants of a link is equivalent to the indeterminacy in regarding it as the closure of a string link.

In the unit disk D^2, we chose a point $e \in \partial D$ and loops $\alpha_1, ..., \alpha_n$ as illustrated in Figure 2.1. For an n–component string link $L = K_1 \cup \cdots \cup K_n$ in $D^2 \times [0,1]$ with $\partial K_j = \{(p_j, 0), (p_j, 1)\}$ $(j = 1, ..., n)$, set $Y = (D^2 \times [0,1]) \setminus L$, $Y_0 = (D^2 \times \{0\}) \setminus L$, and $Y_1 = (D^2 \times \{1\}) \setminus L$. We may assume that each $\pi_1(Y_t)$ $(t \in \{0,1\})$ with base point (e,t) is the free group $F(n)$ on generators $\alpha_1, ..., \alpha_n$. We denote the image of α_j in the lower central series quotient $F(n)/F(n)_q$ again by α_j. By Stallings' theorem [23], the inclusions $i_t : Y_t \longrightarrow Y$ induce isomorphisms $(i_t)_* : \pi_1(Y_t)/\pi_1(Y_t)_q \longrightarrow \pi_1(Y)/\pi_1(Y)_q$ for any positive integer q. Hence the induced map $(i_1)_*^{-1} \circ (i_0)_*$ is an automorphism of $F(n)/F(n)_q$ and sends each α_j to a conjugate $l_j \alpha_j l_j^{-1}$ of α_j, where l_j is the *longitude* of K_j defined as follows. Let γ_j be a zero framed parallel of K_j such that the endpoints $(c_j, t) \in D^2 \times \{t\}$ $(t = 0, 1)$ lie on the x–axis in $\mathbb{R}^2 \times \{t\}$. The longitude $l_j \in F(n)/F(n)_q$ is an element represented by the union of the arc γ_j and the segments $e \times [0,1]$, $c_j e \times \{0,1\}$ under $(i_1)_*^{-1}$. The coefficient $\mu_L(i_1 i_2 ... i_{k-1} j)$ $(k \leq q)$ of $X_{i_1} \cdots X_{i_{k-1}}$ in the Magnus expansion $E(l_j)$ is well-defined invariant of L, and it is called a *Milnor μ–invariant* of length k. It is known that Milnor μ–invariants of length k are finite type invariants of degree $k-1$ for string links [2, 13].

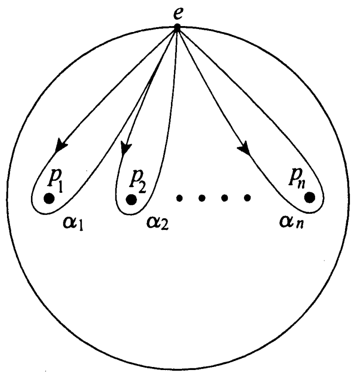

FIGURE 2.1

Convention 2.1. As said above, each Milnor invariant $\mu(I)$ for n-string links is indexed by a sequence I of *possibly repeating* integers in $\{1, ..., n\}$. In the following, when denoting indices of Milnor invariants, we will always let *distinct* letters denote *distinct* integers, unless otherwise specified. For example, $\mu(iijk)$ $(1 \leq i, j, k \leq n)$ stands for all Milnor invariants $\mu(iijk)$ with $i, j, k \in \{1, ..., n\}$ pairwise distincts.

2.3. **Closure invariants.** Given an n-string link L and a sequence $I = i_1 i_2 \cdots i_m$ of m elements in $\{1, ..., n, \overline{1}, ..., \overline{n}\}$, we will construct in the next subsection an oriented knot $K(L; I)$ in S^3 as a closure of L with respect to I. Roughly speaking, we build the knot in S^3 by connecting the endpoints of the i_jth components of L ($j = 1, ..., m$) so that, when running along the knot following the orientation, we meet these components in the order $i_1, i_2, \cdots, i_m$. Indices contained in $\{1, ..., n\}$, resp. in $\{\overline{1}, ..., \overline{n}\}$, correspond to components whose orientation agree, resp. disagree, with the orientation of the knot. If some index appears more than one in I, then we properly take parallels of the corresponding component of L.

2.3.1. *Definition of the knot $K(L; I)$ for a sequence I without repetition.* Let $I = i_1 i_2 \cdots i_m$ be a sequence of m elements in $\{1, ..., n, \overline{1}, ..., \overline{n}\}$ without repeated number, i.e., for each $i = 1, ..., n$, the number of times that i *or* $\overline{i}$ appears in I is at most one. Let $L = K_1 \cup ... \cup K_n$ be an n-string link in $D^2 \times [0, 1] \subset S^3$.

Suppose that $\partial K_i = p_i \times \{0, 1\} \subset D^2 \times \{0, 1\}$. For each I, we choose a tangle T_I in $S^3 \setminus (D^2 \times [0, 1])$ as follows:

- If i_k and i_{k+1} are in $\{1, ..., n\}$ then connect $p_{i_k} \times \{1\}$ and $p_{i_{k+1}} \times \{0\}$ in $S^3 \setminus (D^2 \times [0, 1])$.
- If i_k is in $\{1, ..., n\}$ and i_{k+1} is in $\{\overline{1}, ..., \overline{n}\}$ then connect $p_{i_k} \times \{1\}$ and $p_{\overline{i_{k+1}}} \times \{1\}$ in $S^3 \setminus (D^2 \times [0, 1])$.
- If i_k and i_{k+1} are in $\{\overline{1}, ..., \overline{n}\}$ then connect $p_{\overline{i_k}} \times \{0\}$ and $p_{\overline{i_{k+1}}} \times \{1\}$ in $S^3 \setminus (D^2 \times [0, 1])$.
- If i_k is in $\{\overline{1}, ..., \overline{n}\}$ and i_{k+1} is in $\{1, ..., n\}$ then connect $p_{\overline{i_k}} \times \{0\}$ and $p_{i_{k+1}} \times \{0\}$ in $S^3 \setminus (D^2 \times [0, 1])$.

Here we implicitely mean that $\overline{\overline{i}} = i$ and $i_{m+1} = i_1$ in our notation. Let L_I be the m-string link obtained from L by deleting all components K_j of L such that neither j nor $\overline{j}$ appears in I. Then we have a knot $K(L; I) := L_I \cup T_I$ in S^3. See Figure 2.2 for an example. For each I, we choose T_I so that $K(\mathbf{1}_n; I)$ is the trivial knot. While there are several choices of T_I tangles for each I, we choose one and fix it.

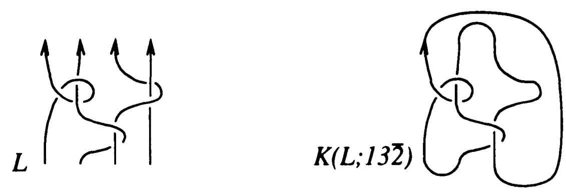

FIGURE 2.2

2.3.2. *Definition of the knot $K(L; I)$ for an arbitrary sequence I.* Let $L = K_1 \cup \cdots \cup K_n$ be an n-string link. Let $I = i_1 i_2 \cdots i_m$ be a sequence of m elements of $\{1, ..., n, \overline{1}, ..., \overline{n}\}$, where for each number $i (= 1, ..., n)$, the number of times that i or $\overline{i}$ appears in I is r_i. Let $m = \sum_i r_i$. Denote by $D_I(L)$ the m-string link obtained from L as follows:

- Replace each string K_i by r_i zero-framed parallel copies of it, labeled from $K_{(i,1)}$ to $K_{(i,r_i)}$ according to the natural order induced by the orientation of the diametral axis in D^2. If $r_i = 0$ for some index i, simply delete K_i.
- Let $D_I(L) = K'_1 \cup \cdots \cup K'_m$ be the m-string link $\bigcup_{i,j} K_{(i,j)}$ with the order induced by the lexicographic order of the index (i,j). This ordering defines a bijection

$$\varphi : \{(i,j) \mid 1 \leq i \leq n, 1 \leq j \leq r_i\} \to \{1, ..., m\}.$$

We also define a sequence $D(I)$ of elements of $\{1, ..., m\}$ without repeated number as follows. First, consider a sequence of elements of $\{(i,j); 1 \leq i \leq n, 1 \leq j \leq r_i\}$ by replacing each number i in I with $(i,1), ..., (i,r_i)$ in this order. For example if $I = 12\overline{2}31$, we obtain the sequence $(1,1),(2,1),\overline{(2,2)},(3,1),(1,2)$. Next replace each term (i,j) of this sequence with $\varphi((i,j))$. Hence we have $D(12\overline{2}31) = 13\overline{4}52$. Since $D(I)$ does not contain repeated number, we have a closure $K(D_I(L); D(I))$ of L with respect to the sequence $D(I)$. We call the knot $K(D_I(L); D(I))$ the *closure knot with respect to I*.

It is not hard to show the following proposition.

Proposition 2.2. *Let I be a sequence of elements in $\{1, ..., n, \overline{1}, ..., \overline{n}\}$, and let v_m be a finite type invariant of degree m for knots. Then the assignement $L \mapsto v_m(K(D_I(L); D(I)))$ defines a finite type invariant of degree m for n-string links.*

Convention 2.3. Let v_m be a finite type invariant of degree $m(\geq 2)$ for knots. For an n-string link L and a sequence $I = i_1 i_2 \cdots i_m$ of m elements of $\{1, ..., n, \overline{1}, ..., \overline{n}\}$, we denote $v_m(K(D_I(L); D(I)))$ by $v_m(L; I)$ or $v_m(D_I(L); D(I))$. For example, we denote $P_0^{(m)}(K(D_I(L); D(I)))$ and $a_m(K(D_I(L); D(I)))$ by $P_0^{(m)}(L; I)$ and $a_m(L; I)$ respectively. We call $P_0^{(m)}(L; I)$ and $a_m(L; I)$ the *$P_0^{(m)}$-closure invariant* and the *a_m-closure invariant* respectively.

3. C_k-MOVES AND FINITE TYPE INVARIANTS

3.1. The Goussarov-Habiro Conjecture.

Goussarov and Habiro showed independently the following.

Theorem 3.1 ([4, 6]). *Two knots (1-string links) cannot be distinguished by any finite type invariant of degree $\leq k$ if and only if they are C_k-equivalent.*

It is known that the 'if' part of the statement holds for links as well, but explicit examples show that the 'only if' part of Theorem 3.1 does not hold for links in general, see [6, §7.2].

However, Theorem 3.1 may generalize to *string* links.

Conjecture (Goussarov-Habiro ; [4, 6]). *Two string links of the same number of components share all finite type invariant of degree $\leq k-1$ if and only if they are C_k-equivalent.*

As in the link case, the 'if' part of the conjecture is always true. The 'only if' part is also true for $k = 1$ (in which case the statement is vacuous) and $k = 2$; the only finite type string link invariant of degree 1 is the linking number, which is known to classify string links up to C_2-equivalence [21]. (Note that this actually also applies to links). The Goussarov-Habiro conjecture was (essentially) proved for $k = 3$ by the first author in [15]. Massuyeau gave a proof for $k = 4$, but it is mostly based on algebraic arguments and thus does not provide any information about the corresponding finite type invariants [14]. In [16], we classify n-string links up to C_k-move for $k \leq 5$, by explicitly giving a complete set of low degree finite type invariants. In addition to Milnor invariants, these include several closure invariants of string links. In the next subsection, we give the statements of these results. As a consequence, we show that the Goussarov-Habiro Conjecture is true for $k \leq 5$.

3.2. **Invariants of degree ≤ 4.** In this subsection, we give a C_k-classification of string links for $k \leq 5$. While the statemants here look different from the statements in [16], they are essentially the same (we just use a different notation for closure invariants).

Recall that there is essentially only one finite type knot invariant of degree 2, namely a_2, and that there is essentially only one finite type knot invariant of degree 3, namely $P_0^{(3)}$. There are essentially two linearly independent finite type knot invariants of degree 4, namely a_4 and $P_0^{(4)}$. We will use these knot invariants to define a number of finite type string links invariants of degree ≤ 4 by using some closure. These various invariants, together with Milnor invariants of length ≤ 5, give the following classification of n-string links up to C_k-equivalence for $k \leq 5$.

Theorem 3.2 ([15]). *Let $L, L' \in \mathcal{SL}(n)$. Then the following assertions are mutually equivalent:*

(1) *L and L' are C_3-equivalent,*
(2) *L and L' share all finite type invariants of degree ≤ 2,*
(3) $a_2(L;i) = a_2(L';i)$ $(1 \leq i \leq n)$,
$a_2(L;i\bar{j}) = a_2(L';i\bar{j})$ $(1 \leq i < j \leq n)$,
$\mu_L(ij) = \mu_{L'}(ij)$ $(1 \leq i < j \leq n)$ *and*
$\mu_L(ijk) = \mu_{L'}(ijk)$ $(1 \leq i < j < k \leq n)$.

Theorem 3.3 ([16]). *Let L, $L' \in \mathcal{SL}(n)$. Then the following assertions are mutually equivalent:*

(1) *L and L' are C_4-equivalent,*
(2) *L and L' share all finite type invariants of degree ≤ 3,*
(3) *L and L' share all finite type invariants of degree ≤ 2, and*
$P_0^{(3)}(L;i) = P_0^{(3)}(L';i)$ $(1 \leq i \leq n)$,
$P_0^{(3)}(L;i\bar{j}) = P_0^{(3)}(L';i\bar{j})$ $(1 \leq i < j \leq n)$
$P_0^{(3)}(L;ik\bar{j}) = P_0^{(3)}(L';ik\bar{j})$ $(1 \leq i < j < k \leq n)$,
$\mu_L(iijj) = \mu_{L'}(iijj)$ $(1 \leq i < j \leq n)$,
$\mu_L(ijkl) = \mu_{L'}(ijkl)$ $(1 \leq i, j < k < l \leq n)$ *and*
$\mu_L(ijkk) = \mu_{L'}(ijkk)$ $(1 \leq i, j, k \leq n,\ i < j)$.

Theorem 3.4 ([16]). *Let L, $L' \in \mathcal{SL}(n)$. Then the following assertions are equivalent:*

(1) *L and L' are C_5-equivalent,*

(2) *L and L' share all finite type invariants of degree ≤ 4,*
(3) *L and L' share all finite type invariants of degree ≤ 3, and*
$a_4(L;i) = a_4(L';i)$, $P_0^{(4)}(L;i) = P_0^{(4)}(L';i)$ $(1 \leq i \leq n)$,
$a_4(L;i\bar{j}) = a_4(L';i\bar{j})$, $P_0^{(4)}(L;i\bar{j}) = P_0^{(4)}(L';i\bar{j})$,
$a_4(L;ii\bar{j}) = a_4(L';ii\bar{j})$, $P_0^{(4)}(L;ii\bar{j}) = P_0^{(4)}(L';ii\bar{j})$,
$P_0^{(4)}(K(L;\overline{ijj})) = P_0^{(4)}(K(L';\overline{ijj}))$ $(1 \leq i < j \leq n)$,
$a_4(L;i\overline{jk}) = a_4(L';i\overline{jk})$, $P_0^{(4)}(L;i\overline{jk}) = P_0^{(4)}(L';i\overline{jk})$,
$a_4(L;i\bar{k}j) = a_4(L';i\bar{k}j)$, $P_0^{(4)}(L;i\bar{k}j) = P_0^{(4)}(L';i\bar{k}j)$,
$a_4(L;ik\bar{j}) = a_4(L';ik\bar{j})$, $P_0^{(4)}(L;ik\bar{j}) = P_0^{(4)}(L';ik\bar{j})$,
$P_0^{(4)}(L;i\bar{j}k) = P_0^{(4)}(L';i\bar{j}k)$ $(1 \leq i < j < k \leq n)$,
$P_0^{(4)}(L;i\bar{j}k\bar{l}) = P_0^{(4)}(L';i\bar{j}k\bar{l})$, $P_0^{(4)}(L;i\bar{j}l\bar{k}) = P_0^{(4)}(L';i\bar{j}l\bar{k})$,
$P_0^{(4)}(L;i\bar{k}j\bar{l}) = P_0^{(4)}(L';i\bar{k}j\bar{l})$ $(1 \leq i < j < k < l \leq n)$,
$\mu_L(ijklm) = \mu_{L'}(ijklm)$ $(1 \leq i,j,k < l < m \leq n)$,
$\mu_L(iiijk) = \mu_{L'}(iiijk)$, $\mu_L(ijjkk) = \mu_{L'}(ijjkk)$,
$\mu_L(jikll) = \mu_{L'}(jikll)$ $(1 \leq i,j,k,l \leq n,\ j < k)$.

Remark 3.5. A complete set of finite type link invariant of degree ≤ 3 has been computed in [10] using weight systems and chord diagrams. For 2-component links, this has been done for degree ≤ 4 invariants in [11]. All invariants are given by coefficients of the Conway and HOMFLYPT polynomials of sublinks.

4. Milnor invariants and $P_0^{(m)}$-closure invariants

We start by expressing Milnor's link homotopy invariants, i.e., Milnor invariants $\mu(I)$ with a sequence I without repeated number, in terms of the closure invariants defined in Subsection 2.3.

Theorem 4.1 ([17]). *Let $m \geq 2$. Let L be a C_m-trivial n-string link $(m+1 \leq n)$. Let I be a sequence of $m+1$ elements of $\{1,...,n\}$ without repeated number. Then*

$$\mu_L(I) = \frac{\pm 1}{m!2^m} \sum_{J \subset I, J \neq \emptyset} (-1)^{m-|J|} P_0^{(m)}(L;J),$$

where the sum runs over all nonempty subsequences J of I.

Remark 4.2. (1) By [6], the fact that L is C_m-trivial implies that $\mu_L(I) = 0$ for any sequence I of length $|I| \leq m$.
(2) Any link-homotopically trivial Brunnian n-string link is C_n-trivial [8, 20], and any Brunnian n-string link whose Milnor invariants of length $\leq n+1$ vanish is C_{n+1}-trivial [16]. Since a Brunnian n-string link whose Milnor invariants with length $\leq n$ vanish is link-homotopically trivial [18], for $m = n+1$ or n, a Brunnian n-string link whose Milnor invariants with length $\leq m$ vanish is C_m-trivial. Moreover, any Brunnian n-string link is C_{n-1}-trivial [7, 20] and has vanishing Milnor invariants with length $\leq n-1$, so this holds for $m = n-1$ as well.
(3) Since there exists no degree one invariant of knots, such a formula does not hold for the linking number, hence the assumption $m \geq 2$ is needed. In order to give such

a formula one should consider 'closure links', that is more general closure operations on string links that can produce links with several components.

By combining [19, Thm. 7] and Theorem 4.1, we have the following theorem.

Theorem 4.3 ([17]). *Let $m \geq 2$. Let L be a C_m-trivial n-string link. Let I be a sequence of $m+1$ elements of $\{1, ..., n\}$. Then*

$$\mu_L(I) = \mu_{D_I(L)}(D(I)) = \frac{\pm 1}{m!2^m} \sum_{J \subset D(I), J \neq \emptyset} (-1)^{m-|J|} P_0^{(m)}(D_I(L); J),$$

where the sum runs over all nonempty subsequences J of $D(I)$.

K. Habiro has pointed out the following remark.

Remark 4.4. It is not hard to see that the 6-string link L illustrated in Figure 4.1 is C_5-trivial and satisfies $\mu_L(123456) = \pm 1$. By Theorem 4.1, $\mu_L(123456)$ can be expressed as a linear combination of $P_0^{(5)}$-closure invariants of L. (By applying the theorem, we have $\mu_L(123456) = (\pm 1/5!2^5) P_0^{(5)}(L; 123456)$.) In contrast, since a_5 of knots always vanish, it is impossible to express $\mu_L(123456)$ by any linear combination of a_5-closure invariants of L. Moreover we notice that L is equivalent to $\mathbf{1}_6$ up to *doubled-delta move*, which is a local move on links defined by Naik and Stanford [22]. Hence any closure knot, and more generally any closure link (see Remark 4.2(3)) obtained from L is equivalent to a trivial knot or link up to doubled-delta moves. Since the doubled-delta move preserves the Alexander invariant, the Conway polynomial of any closure link obtained from L vanishes.

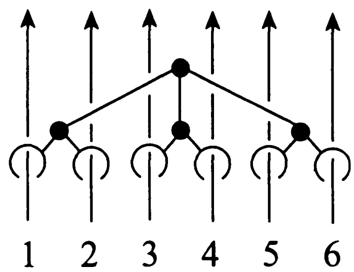

FIGURE 4.1

5. CLASPERS AND $P_0^{(m)}$-CLOSURE INVARIANTS

5.1. **Claspers.** For a general definition of claspers, we refer the reader to [6]. Let L be a (string) link. A surface G embedded in $D^2 \times (0,1)$ is called a *graph clasper* for L if it satisfies the following three conditions:

(1) G is decomposed into disks and bands, called *edges*, each of which connects two distinct disks.
(2) The disks have either 1 or 3 incident edges, and are called *leaves* or *nodes* respectively.
(3) G intersects L transversely, and the intersections are contained in the union of the interiors of the leaves.

In particular, if a connected graph clasper G is simply connected, we call it a *tree clasper.*

A graph clasper for a (string) link L is *simple* if each of its leaves intersects L at one point. The degree of a connected graph clasper G is defined as half of the number of nodes and leaves. We call a degree k connected graph clasper a *C_k-graph.* A tree clasper of degree k is called a *C_k-tree.*

Given a graph clasper G for a (string) link L, there is a procedure to construct a framed link, in a regular neighbourhood of G. There is thus a notion of *surgery along* G, which is defined as surgery along the corresponding framed link. In particular, surgery along a simple C_k-tree is a local move as illustrated in Figure 5.1, which is equivalent to a C_k-move as defined in Section 1 (Figure 1.1).

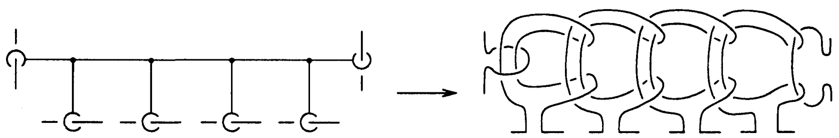

FIGURE 5.1. Surgery along a simple C_5-tree.

The C_k-equivalence (as defined in Section 1) coincides with the equivalence relation on string links generated by surgeries along C_k-graphs and isotopies. In particular, it is known that two links are C_k-equivalent if and only if they are related by surgery along simple C_k-trees [6, Thm. 3.17].

For $k \geq 3$, a C_k-tree G having the shape of the tree clasper in Figure 5.1 is called a *linear* C_k-tree. The left-most and right-most leaves of G in Figure 5.1 are called the *ends* of G, and the remaining $(k-1)$ leaves are called the *internal leaves* of G.

Suppose that the two ends of a linear C_k-tree are denoted by f and f'. Let $\mathcal{S}$ be a nonempty subset of the set of all internal leaves of T. We have a labeling from 1 to $|\mathcal{S}|$ of the leaves in $\mathcal{S}$ by travelling along the boundary of the disk[1] T from f to f' so that all leaves are visited. We call this labeling the *linear labeling of $\mathcal{S}$, from f to f'.*

5.2. **Generators of $\mathcal{SL}_m(n)/C_{m+1}$.** Let $m \geq 3$ be an integer. In this section we find generators for the abelian group $\mathcal{SL}_m(n)/C_{m+1}$ and show that for each of these generators, there is a $P_0^{(m)}$-closure invariant which detects it.

For a simple tree clasper Γ for a string link, let $r_i(\Gamma)$ denote the number of leaves intersecting the ith component of the string link.

Let $L \in \mathcal{SL}_m(n)$ be a C_m-trivial n-string link. By *Calculus of Claspers* [16, Lem.3.2] and the *AS and IHX relations* [16, Lem.3.3], L is C_{m+1}-equivalent to a product $\prod T_i$ of n-string links $T_1, ..., T_l$, where each T_k is obtained from $\mathbf{1}_n$ by surgery along a simple linear C_m-tree Γ_k. Actually, by the IHX relation we may assume that each Γ_k satisfies one of the following;

(1) all leaves of Γ_k intersect a single component of $\mathbf{1}_n$,
(2) $|\{i \mid r_i(\Gamma_k) = 1\}| \geq 2$, and the ends intersect the pth and qth components of $\mathbf{1}_n$, where $p = \min\{i \mid r_i(\Gamma_k) = 1\}$ and $q = \min\{i \mid r_i(\Gamma_k) = 1,\ i \neq p\}$,

[1]Recall that a clasper is an embedded surface: in particular, since T is a tree clasper, the underlying surface is isotopic to a disk.

(3) $r_i(\Gamma_k) = 2$ for some i, $|\{i \mid r_i(\Gamma_k) = 1\}| < 2$, and the ends intersect the pth component of $\mathbf{1}_n$, where $p = \min\{i \mid r_i(\Gamma_k) = 2\}$,
(4) Γ_k is not of type (1), $r_i(\Gamma_k) \neq 2$ for any i, $|\{i \mid r_i(\Gamma_k) = 1\}| < 2$, and the ends intersect the pth component of $\mathbf{1}_n$, where $(r_p(\Gamma_k), p)$ is the minimum among $\{(r_i(\Gamma_k), i) \mid i = 1, ..., n,\ r_i(\Gamma_k) \geq 3\}$ with respect to the lexicographic order.

This implies that $\mathcal{SL}_m(n)/C_{m+1}$ is generated by all string links obtained from $\mathbf{1}_n$ by surgery along a C_m-tree of one of the 4 types above.

Let us reduce the number of generators of type (4). Let $\mathcal{T}_p$ be the set of linear C_m-trees of type (4) with ends intersecting the pth component of $\mathbf{1}_n$. Each tree in $\mathcal{T}_n$ has a unique leaf not intersecting the nth component of $\mathbf{1}_n$. By [16, Lem.3.6], the case reduces to trees of type (3). Hence we may assume that $p \neq n$. By the IHX relation, we may assume that the two ends are the *'top'*, resp. *'bottom', leaves on the pth component of* $\mathbf{1}_n$, which are defined as the last, resp. first, leaf we meet while traveling along this component from the initial point to the terminal point. For a C_m-tree $\Gamma \in \mathcal{T}_p$ with top end f and bottom end f', we consider the linear labeling (from 1 to $m-1$) of the set of all internal leaves of Γ, from f' to f (see Section 5.1). Suppose that while traveling along the pth component from f' to f, we meet s leaves labeled by $i_1, ..., i_s \in \{1, ..., m-1\}$ in this order. We say that Γ is *flat (on the pth component of* $\mathbf{1}_n$*)* if $i_1 < i_2 < \cdots < i_s$. Let $\mathcal{F}_p$ be the set of flat trees in $\mathcal{T}_p$.

Define $\mathcal{F}_p^0$ as set of C_k-trees in $\mathcal{F}_p$ which do not contain a *fork*. Here we say that a tree clasper T for $\mathbf{1}_n$ contains a fork if there exists a 3-ball that intersects $\mathbf{1}_n \cup T$ as represented in Figure 5.2

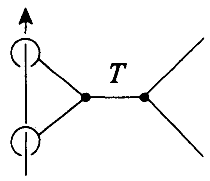

FIGURE 5.2

Proposition 5.1 ([17]). *For an integer $m \geq 3$, $\mathcal{SL}_m(n)/C_{m+1}$ is generated by string links obtained from $\mathbf{1}_n$ by surgery along linear trees of type* (1), (2), (3) *or in $\mathcal{F}_p^0$ ($p = 1, ..., n-1$).*

The abelian group $\mathcal{SL}_m(n)/C_{m+1}$ can be decomposed into a direct sum $G_1 \oplus G_2$, where G_1 (resp. G_2) is the subgroup generated by string links obtained from $\mathbf{1}_n$ surgery along a linear C_m-tree of type (1) (resp. of type (2), (3) or in $\mathcal{F}_p^0$ ($p = 1, ..., n-1$)). By the Goussarov-Habiro Theorem [4, 6], G_1 is classified by finite type invariants. For the group G_2, we have the following

Theorem 5.2 ([17]). *Let $m \geq 3$ be an integer. For any simple linear C_m-tree Γ for $\mathbf{1}_n$ of type* (2), (3) *or in $\mathcal{F}_p^0$ ($p = 1, ..., n-1$), there is a sequence I of elements of $\{1, ..., n\}$ such that $P_0^{(m)}((\mathbf{1}_n)_\Gamma; I) = \pm m! 2^m$. Hence $(\mathbf{1}_n)_\Gamma$ has infinite order in $\mathcal{SL}_m(n)/C_{m+1}$.*

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